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RAMSEY NUMBERS FOR TREES OF SMALL MAXIMUM DEGREE

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For a tree T we write t_1 and t_2, $t_2 \geq t_1$, for the sizes of the vertex classes of T as a bipartite graph. It is shown that for T with maximum degree $o(t_2)$, the obvious lower bound for the Ramsey number $R(T,T)$ of $\max\{2t_1+t_2, 2t_2\}-1$ is asymptotically the correct value for $R(T,T)$.

1. Introduction

The *Ramsey number* $R(G,G)$ of a graph G is defined to be the smallest integer n such that the following holds. In every colouring of the edges of the complete graph K_n with two colours, there exists a copy of G whose edges are all the same colour. In this paper we study the Ramsey number for trees T. It is easy to see that if the sizes of the vertex classes of T as a bipartite graph are t_1 and t_2, where $2t_1 \geq t_2 \geq t_1 \geq 2$, then $R(T,T) \geq 2t_1+t_2-1$ since the following "canonical" colouring of $K_{2t_1+t_2-2}$ has no monochromatic copy of T. We just partition the vertex set into a class C_1 of size t_1+t_2-1 and a class C_2 of size t_1-1, colour red all edges that join a vertex in C_1 to one in C_2, and colour the rest blue. For the case in which $2t_1 < t_2$, a similar colouring using class sizes t_2-1 and t_2-1 shows that $R(T,T) \geq 2t_2-1$.

It has been conjectured by Burr [1] (see also [2]) that the above lower bounds give the correct value of $R(T,T)$. This has been verified for various special types of trees ([3], [4]; see also [7]), but Grossman, Harary and

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Klawe [5] proved that in general the conjecture is not true: for some "double-star" S the Ramsey number $R(S,S)$ is by one (!) larger than its anticipated value. However, it is still possible that, although the lower bounds obtained by the canonical colourings do not give the correct value of $R(T,T)$, they approximate it very closely. Our main result states that this is indeed the case for each tree T with moderate maximum degree.

Theorem 1. *Let $\eta > 0$ be given. Then there exist $N = N(\eta)$ and $\delta = \delta(\eta)$ such that the following holds. For every $t_1 \le t_2 \in \mathbb{Z}$ with $t_2 \ge N$, and every tree T with bipartition $V_1(T) \cup V_2(T)$, where $|V_1(T)| = t_1$, $|V_2(T)| = t_2$, and with maximum degree $\Delta(T) \le \delta t_2$, we have the following.*

(i) *If $2t_1 \ge t_2$ then $R(T,T) \le (1+\eta)(2t_1 + t_2)$,*
(ii) *If $2t_1 < t_2$ then $R(T,T) \le (1+\eta)(2t_2)$.*

2. The idea of the proof

The proof of Theorem 1 is based on Szemeredi's Regularity Lemma which states that for any two-colouring of the complete graph K_n one can partition its vertices into a moderate number of equal sets, $U_1, \dots, U_s$, $|U_1| = \dots = |U_s| = \hat{n}$, in such a way that almost all of the pairs (U_i, U_j) are "regular", i.e., edges of each of the colour classes are "uniformly" distributed between U_i and U_j. It can also be shown that the bipartite graph induced in one colour by such a regular pair (U_i, U_j) which is dense enough contains all small trees. This fact suggests the following strategy for finding a monochromatic copy of a tree T in a two-coloured complete graph K_n (see also [6], where a similar approach has been used):

- apply the Regularity Lemma to the two-coloured graph K_n,
- choose one of the colours, say blue, such that the graph contains a rich configuration of blue regular pairs,
- divide T into a small subtrees $T_1, \dots, T_t$,
- embed trees $T_1, \dots, T_t$ one by one into blue subgraphs induced by regular pairs.

The above method of embedding the tree T piece by piece works nicely for trees T that have very small maximum degree, say, not larger than $n^{o(1)}$. However, if we allow vertices of T to have larger degrees, then the argument becomes more complicated, because, roughly speaking, in order to keep the number of trees t small, the trees $T_1, \dots, T_t$ must be "tightly packed" in T. More precisely, denote by $(V_1(T), V_2(T))$ the bipartition of T and for $\ell = 1, \dots, t$, let $(V_1(T_\ell), V_2(T_\ell))$, where $V_1(T_\ell) \subseteq V_1(T)$, denote the bipartition of T_ℓ. Then, typically, a decomposition $T_1,\dots,T_k$ of T has the following property: once we decide to embed $T_{\ell'}$ into a regular pair (U_i,U_j) such that vertices from $V_1(T_{\ell'})$ are mapped into U_i, for every other tree $T_{\ell''}$ embedded into (U_i,U_j) we have to map vertices from $V_1(T_{\ell''})$ into U_i as well, i.e., each embedding of trees $T_1,\dots,T_\ell$ into regular pairs must preserve the ordering of the bipartition classes. Hence, if $\alpha = |V_2(T)|/|V_1(T)| > 1$, and for each tree T_ℓ we have, say, $|V_2(T_\ell)|/|V_1(T_\ell)| > (1+\alpha)/2$, then the procedure which packs trees $T_1,\dots,T_k$ into regular pairs (U_i,U_j), where, recall, $|U_i| = |U_j|$, is not very effective. Indeed, all vertices from the larger classes $V_2(T_\ell)$ will be mapped into just one of the sets U_i and U_j, say, U_j, leaving out a fair proportion of the "dropout" vertices of U_i; the edges joining these leftover vertices with U_j will remain unused during the embedding procedure. Thus, we use some results on real-valued functions defined on edges of two-coloured graphs, to replace the partition obtained from the Regularity Lemma by a "non-balanced" partition with partition classes of different sizes which, furthermore, contains a certain special configuration in one of the colours.

The structure of the paper is the following. In the next section we recall the notion of the regular pair and the Regularity Lemma and, most importantly, state without proof the result on the existence of a special class of non-balanced partitions in two-coloured complete graphs, which is the key ingredient of our argument (Theorem 3). Then, we describe a partition procedure which cuts T into small trees $T_1,\dots,T_\ell$ (Theorem 4), and deduce from Theorems 3 and 4 the main result of the paper, Theorem 1. The last three sections of the article are devoted to the proof of Theorem 3. Thus, in Section 6, we introduce a special class of weighted functions on digraphs, and study a special type of a partition of the vertices of a digraph on which such a function is defined (Theorem 8). Then, we use this function to prove Theorem 10 on real-valued functions defined on two-coloured very dense graphs. Finally, in the last section, we show that Theorem 10 implies Theorem 3.

3. Regular pairs and the Regularity Lemma

Let a graph G with n vertices be fixed. For U_1, $U_2 \subset V = V(G)$ with $U_1 \cap U_2 = \emptyset$, we write $E(U_1,U_2) = E_G(U_1,U_2)$ for the set of edges of G that have one end in U_1 and the other in U_2, and $G[U_1,U_2]$ for the subgraph of G with vertex set $U_1 \cup U_2$ and edge set $E_G(U_1,U_2)$. The *density* $d(U_1,U_2)$ of the pair (U_1,U_2) is defined by

$$d(U_1,U_2) = |E(U_1,U_2)|/|U_1||U_2|\,.$$

Suppose $\epsilon>0$. We say that a pair (U_1,U_2) is *ϵ-regular for G* if for all $U_1'\subseteq U_1$, $U_2'\subseteq U_2$ with $|U_1'|\geq\epsilon|U_1|$ and $|U_2'|\geq\epsilon|U_2|$, we have

$$|d(U_1',U_2')-d(U_1,U_2)|<\epsilon\,.$$

Note that if (U_1,U_2) is an ϵ-regular pair for some $0<\epsilon<1/2$, and $W_i\subseteq U_i$, $|W_i|\geq\beta|U_i|$ for $i=1,2$, and some $\beta>0$, then the pair (W_1,W_2) is (ϵ/β)-regular. Notice also that if we colour the edges of the complete graph by two colours, blue and red, then a pair (U_1,U_2) is ϵ-regular in the blue graph if and only if it is ϵ-regular in the red graph.

We say that a partition $\Pi=(U_i)_0^s$ of $V=V(G)$ is *(ϵ,s)-equitable* if $|U_0|\leq\epsilon n$, and $|U_1|=\ldots=|U_s|$. Then, the well-known Szemerédi Regularity Lemma [8] can be stated as follows.

Lemma 2. *Let a real number $\epsilon>0$ and a positive integer s_0 be given. Then there exists a constant $S_0=S_0(\epsilon,s_0)\geq s_0$ such that for any graph G there exists an (ϵ,s)-equitable partition $\Pi=(U_i)_0^s$ of the set of vertices of G, such that $s_0\leq s\leq S_0$, and all but at most $\epsilon\binom{s}{2}$ pairs (U_i,U_j) with $1\leq i<j\leq s$ are ϵ-regular.* ∎

Finally, we state a result on the existence of certain non-balanced partitions in a two-coloured complete graph. This theorem is crucial for our considerations; unfortunately, its proof is long and complicated so we postpone it until Sections 6–8.

Theorem 3. *Let $\epsilon>0$ and α, $1\leq\alpha\leq2$, be given. Then there exist $n_0=n_0(\epsilon)$ and $k_0=k_0(\epsilon)$ such that the following holds for every $n\geq n_0$.*

In every two-colouring of the edges of K_n, there exists a monochromatic subgraph H_α with the following properties:

(i) *$V(H_\alpha)$ is a union of disjoint sets $Y_0\cup\bigcup_{i=1}^k X_i\cup\bigcup_{i=1}^k Y_i$ where $k\leq k_0$,*
(ii) *$|X_i|=\tilde{n}$ for $1\leq i\leq k$ and $|Y_i|=\lceil\alpha\tilde{n}\rceil$ for $0\leq i\leq k$, where $\tilde{n}\geq(1-\epsilon)(2+\alpha)^{-1}n/k$,*
(iii) *(Y_0,X_i) is ϵ-regular of density at least 1/3 for $1\leq i\leq k$,*
(iv) *(X_i,Y_i) is ϵ-regular of density at least 1/3 for $1\leq i\leq k$.*

4. Tree partitions

We say that a tree T is a *rooted* tree if it has a distinguished vertex $r(T)$, the *root* of T. In general for a bipartite graph G we shall write $V_1(G)$ and $V_2(G)$ for the bipartition classes of G. Also for a general graph G and vertex $v\in V(G)$ we use $\Gamma(v)$ to denote the neighbourhood of v in G.

Theorem 4. *Let a positive integer $s \geq 4$ and a tree T be given. Then there exist rooted subtrees $T_1,\dots,T_q$ of T with the following properties.*

(i) $T = \bigcup_{i=1}^{q} T_i$,
(ii) $s/2 \leq |V(T_i)| \leq s$ *for each* $2 \leq i \leq q$,
(iii) $|V(T_1)| \leq s$,
(iv) *for* $2 \leq i \leq q$ *we have that* $V(T_i) \cap \bigcup_{j<i} V(T_j) = \{r(T_i)\}$.

Proof. Let s and T be given as in the statement of the lemma. We shall define subtrees S_i one by one, and in the end re-label them T_i such that Properties (iii) and (iv) are satisfied.

Suppose trees $S_1 \dots S_{i-1}$ have been defined such that the following conditions are satisfied:

(a) the tree T' induced by

$$V(T) \setminus [(V(S_1) \setminus \{r(S_1)\}) \cup \ldots \cup (V(S_{i-1}) \setminus \{r(S_{i-1})\})]$$

has at least one edge,
(b) $s/2 \leq |V(S_j)| \leq s$ for each S_j, $1 \leq j \leq i-1$,
(c) $r(S_{i-1})$ is a vertex of T', and for $1 \leq j \leq i-2$, if $V(S_j) \cap V(T') \neq \emptyset$ then $V(S_j) \cap V(T') = \{r(S_j)\}$.

We now describe how to construct S_i. Let x_0 be any vertex of T' and consider T' to be rooted at x_0. For a vertex x of T' we denote by $T'(x)$ the subtree of T' induced by x together with all vertices whose path to x_0 contains x. Also we write $L_i(T')$ for the set of vertices of T' that are at distance i from x_0 in T'.

Now, if $|V(T')| \leq s$, then we set $S_i = S_q = T'$, choose as $r(S_i) = y$ any vertex of $V(T')$, and stop. Otherwise there is a sequence x_0, $x_1,\dots,x_m$ of vertices such that $x_j \in L_j(T')$ and $|V(T'(x_m))| > s$, but $|V(T'(y))| \leq s$ for every $y \in Y = L_{m+1}(T') \cap \Gamma(x_m)$. If any element y of Y satisfies $|V(T'(y))| \geq s/2$, then we let $S_i = T'(y)$ and $r(S_i) = y$. Otherwise let $U \subseteq Y$ be any subset such that $|\bigcup_{u\in U} V(T'(u))| < s$, but $|V(T'(y)) \cup \bigcup_{u \in U} V(T'(u))| \geq s$ for some $y \in Y \setminus U$. Then we take $S_i = \{x_m\} \cup \bigcup_{u\in U} T'(u)$ and $r(S_i) = x_m$, which is a tree satisfying $s/2 \leq |V(S_i)| \leq s$ as required.

We now note that unless $i = q$, Properties (a)–(c) are still satisfied for $S_1,\dots,S_i$ by construction. If $i = q$ then (a) and (c) are satisfied, $|V(S_q)| \leq s$ and the construction is complete. This completes the initial definition of $S_1,\dots,S_q$.

Now we show that $S_1,\dots,S_q$ can be relabelled $T_1,\dots,T_q$ so that conditions (i)–(iv) are satisfied. Note that (i) and (ii) are satisfied by (a) and (b). We know by (c) that $|V(S_i) \cap V(S_j)| \leq 1$ for all $i \neq j$, and if $|V(S_i) \cap V(S_j)| = 1$

and $j > i$, then the common vertex is the root of S_i. We construct a directed graph D with vertex set $S_1, \dots, S_q$, where (S_i, S_j) is an arc if and only if $r(S_i)$ is a vertex of S_j and $j > i$ is the smallest index such that this is true. By (c), each S_i, $i < q$, has precisely one arc (S_i, S_j) leaving it. Therefore relabelling S_q as T_1, its k_1 in-neighbours as $T_2, \dots, T_{k_1+1}$, the k_2 in-neighbours of these by $T_{k_1+2}, \dots, T_{k_1+k_2+1}$, and so on, guarantees that Properties (iii) and (iv) hold. ∎

5. Proof of Theorem 1

Proof of Theorem 1. Let $\eta > 0$ be given. Let $\epsilon = \min\{10^{-6}, (\eta/200)^2\}$, let $n_0(\epsilon)$ and $k_0 = k_0(\epsilon)$ be defined as in Theorem 3, and let $\delta = \epsilon^2/100k_0^2$ and $N = \max\{n_0(\epsilon), 24k_0/\epsilon^2\}$. Let T be a tree with $t_1 = |V_1(T)|$ and $t_2 = |V_2(T)|$, where $t_1 \le t_2$ and $t_2 \ge N$, and suppose that T has maximum degree at most δt_2.

First we note that Part (2) of the theorem follows from Part (1) applied with parameter $\eta/2$. To see this, note that if $2t_1 < t_2$ then T is a subtree of another tree T' with $|V_2(T')| = t_2$ and $|V_1(T')| = \lceil t_2/2 \rceil$, with maximum degree at most δt_2. Therefore $R(T,T) \le R(T',T') \le (1+\eta/2)(2t_2+1) \le (1+\eta)2t_2$ by Part (1). Therefore we may assume that $2t_1 \ge t_2$.

Let $\alpha = t_2/t_1$, so we have $1 \le \alpha \le 2$, and let $n = \lfloor (1+\eta)(1+2/\alpha)t_2 \rfloor$. Then note that $n > n_0(\epsilon)$, so that Theorem 3 can be applied to K_n with parameter ϵ. Note also that

$$t_2 < \alpha n/(\alpha+2). \tag{1}$$

We shall show that every 2-colouring of the edges of K_n contains a monochromatic copy of T. Our strategy will be to cut T into small subtrees using Theorem 4, and embed them one-by-one into the monochromatic subgraph H_α of K_n whose existence is guaranteed by Theorem 3.

Let H_α, $k \le k_0(\epsilon)$, and $\tilde{n}$ be as in Theorem 3 applied to K_n with parameter ϵ, so in particular

$$\tilde{n} \ge \frac{(1-\epsilon)n}{(2+\alpha)k}. \tag{2}$$

We apply Theorem 4 to T with $s=\lfloor\epsilon\tilde{n}\rfloor$ to obtain subtrees $T_1,\dots,T_q$. Then, from Theorem 4(ii) and (1), we have

$$q \le \frac{t_1+t_2}{s/2-1}+1 < 3\frac{t_1+t_2}{s} < 3\frac{2\alpha n}{(\alpha+2)s} \tag{3}$$
$$< 3\frac{2\alpha}{\alpha+2}\frac{k\tilde{n}(\alpha+2)}{(1-\epsilon)s} \le \frac{6k\tilde{n}}{(1-\epsilon)\lfloor\epsilon\tilde{n}\rfloor} < \frac{12k}{\epsilon}\,.$$

Our embedding will place each subtree (except possibly for a few of its vertices) into one of the regular pairs (X_i,Y_i) of the graph H_α. Our first step therefore is to decide in advance into which pair each subtree T_ℓ should be embedded, so that they will all fit into the available space. Our embedding will succeed in placing each subtree into the correct pair, except for a few subtrees which we shall call *exceptional*. To accommodate these exceptional subtrees we shall also reserve $k_1=100\epsilon k$ pairs (X_i,Y_i) before beginning the embedding. Below we use the notation $[q]=\{1,\dots,q\}$.

Lemma 5. *There exists a function $\gamma:[q]\to\{k_1+1,\dots,k\}$ such that for each $i\in\{k_1+1,\dots,k\}$ we have*

$$\sum_{\ell:\gamma(\ell)=i}|V_1(T_\ell)| < (1-100\epsilon)\tilde{n} \tag{4}$$

and

$$\sum_{\ell:\gamma(\ell)=i}|V_2(T_\ell)| < (1-100\epsilon)\alpha\tilde{n}. \tag{5}$$

Proof. Let $A\subseteq[q]$ and $\gamma:A\to\{k_1+1,\dots,k\}$ be such that:

(a) for each $i\in\{k_1+1,\dots,k\}$ both inequalities (4) and (5) hold,
(b) there are no A' and $\gamma':A'\to\{k_1+1,\dots,k\}$ with $|A'|>|A|$ for which property (a) holds,
(c) A and γ minimize the value of the function

$$f(\gamma,A)=\sum_{i=k_1+1}^{k}\left[\left(\sum_{\ell:\gamma(\ell)=i}|V_2(T_\ell)|\Big/\sum_{\ell:\gamma(\ell)=i}|V_1(T_\ell)|\right)-\alpha\right]^2$$

over all pairs (A'',γ'') which fulfill (a) and (b).

Thus, it is enough to show that $A=[q]$. Suppose that it is not the case. Note first that then, for every $i\in\{k_1+1,\dots,k\}$, we must have either

$$\sum_{\ell:\gamma(\ell)=i}|V_1(T_\ell)| > (1-101\epsilon)\tilde{n}, \tag{6}$$

or

$$\sum_{\ell:\gamma(\ell)=i} |V_2(T_\ell)| > (1-101\epsilon)\alpha\tilde{n}. \tag{7}$$

Indeed, since $|V(T_\ell)| \le \epsilon\tilde{n}$ for every $\ell \in A$, if both above inequalities would fail for some $i_0 \in \{k_1+1,\dots,k\}$ one could extend the domain A of γ to A' by choosing any $q_0 \in [q]\setminus A$ and putting $\gamma(q_0)=i_0$, contradicting the choice of A.

Furthermore, using (3) and the definition of N we find

$$\sum_{\ell=1}^{q} |V_1(T_\ell)| \le t_1 + q < t_1 + 12k/\epsilon < (1+\epsilon)t_1.$$

Therefore from (2) and the definition of n, and the fact that $\eta > 100\sqrt{\epsilon}$ and $\epsilon < 10^{-6}$, we infer that there exists $i_1 \in \{k_1+1,\dots,k\}$ such that

$$\begin{aligned}\sum_{\ell:\gamma(\ell)=i_1} |V_1(T_\ell)| &\le \frac{t_1(1+\epsilon)}{k-k_1} \le \frac{1+2\epsilon}{(1-100\epsilon)(1+\eta)(2+\alpha)}\frac{n}{k}\\ &\le (1-50\sqrt{\epsilon})\tilde{n},\end{aligned} \tag{8}$$

i.e., for i_1, from (6) and (7) the inequality (6) is true. Similarly, for some $i_2 \in \{k_1+1,\dots,k\}$, we must have

$$\sum_{\ell:\gamma(\ell)=i_2} |V_2(T_\ell)| \le (1-50\sqrt{\epsilon})\alpha\tilde{n}, \tag{9}$$

and so for i_2 (7) holds.

One can use (7) and (8) to find a family of trees $T_{\ell'_1},\dots,T_{\ell'_r}$ such that for every $t=1,\dots,r$ we have $\gamma(\ell'_t)=i_1$,

$$|V_2(T_{\ell'_t})|/|V_1(T_{\ell'_t})| \ge (1+10\sqrt{\epsilon})\alpha\,, \tag{10}$$

and

$$(10\sqrt{\epsilon}-\epsilon)\alpha\tilde{n} \le \sum_{t=1}^{r} |V_2(T_{\ell'_t})| \le 10\sqrt{\epsilon}\alpha\tilde{n}\,. \tag{11}$$

Again, by the analogous argument, there is a family of trees $T_{\ell''_1},\dots,T_{\ell''_s}$ such that for every $t=1,\dots,s$ we have $\gamma(\ell''_t)=i_2$,

$$|V_2(T_{\ell''_t})|/|V_1(T_{\ell''_t})| \le (1-10\sqrt{\epsilon})\alpha\,, \tag{12}$$

and

$$(10\sqrt{\epsilon}-\epsilon)\tilde{n} \le \sum_{t=1}^{s} |V_1(T_{\ell''_t})| \le 10\sqrt{\epsilon}\tilde{n}\,. \tag{13}$$

Now we replace $\gamma\colon A\to\{k_1{+}1,\dots,k\}$ by $\bar\gamma\colon A\to\{k_1{+}1,\dots,k\}$ by swapping the above families of trees, i.e., by setting $\bar\gamma(\ell'_t)=i_2$, for $t=1,\dots,r$, and $\bar\gamma(\ell''_t)=i_1$, for all $t=1,\dots,s$. Observe that for such $\bar\gamma$ the inequalities (4) and (5) hold. Indeed, for $i=i_1$, the inequalities (8) and (13) give

$$\sum_{\ell:\bar\gamma(\ell)=i_1}|V_1(T_\ell)|\le(1-50\sqrt{\epsilon})\tilde n+10\sqrt{\epsilon}\tilde n<(1-100\epsilon)\tilde n\,.$$

On the other hand, from (11), (12) and (13) we get

$$\sum_{\ell:\bar\gamma(\ell)=i_1}|V_2(T_\ell)|<(1-100\epsilon)\alpha\tilde n-\sum_{t=1}^{r}|V_2(T_{\ell'_t})|$$
$$+(1-10\sqrt{\epsilon})\alpha\sum_{t=1}^{s}|V_1(T_{\ell''_t})|<(1-100\epsilon)\alpha\tilde n\,.$$

One can easily verify (4) and (5) for $i=i_2$ in a similar way.

Now, to complete the proof, it is enough to observe that $f(\bar\gamma,A)<f(\gamma,A)$, contradicting the choice of γ and A. ∎

Our next step is to prepare the subgraph H_α. Recall from Theorem 4 that each subtree T_i may contain vertices that are roots of subtrees T_j where $j>i$. When embedding T_i we need to ensure that these roots of "future" T_j's are embedded into vertices of H_α that have lots of neighbours that have not yet been used in the embedding. Therefore we will select special subsets R_i of X_i for each i, and a special subset R_0 of Y_0, into which these roots can be embedded. In what follows, for a vertex x and a set of vertices S we shall denote by $d_S(x)$ the quantity $|\Gamma(x)\cap S|$.

First we partition off a subset R'_i from each X_i, of size $\lceil 22\epsilon\tilde n\rceil$.

Now, by Theorem 3(iii), for each $i\in[k_1]$ there are at most $\epsilon\alpha\tilde n$ vertices $y\in Y_0$ such that $d_{R'_i}(y)<(1/3-\epsilon)|R'_i|$. Therefore the number of pairs (y,i) with $y\in Y_0$ and $i\in[k_1]$ such that $d_{R'_i}(y)<(1/3-\epsilon)|R'_i|$ is at most $k_1\epsilon\alpha\tilde n$. Let us say that such a y is *bad* for i. Therefore the number of vertices in Y_0 that are bad for at least $2\epsilon k_1$ different values of $i\in[k_1]$ is at most $|Y_0|/2$. We may therefore choose a set $R_0\subset Y_0$ of size at least $|Y_0|/2$ such that every vertex of R_0 is bad for at most $2\epsilon k_1$ values of $i\in[k_1]$.

Now for each $i\in[k]$ we let $R_i\subset R'_i$ be a set of size $\lceil 21\epsilon\tilde n\rceil$ such that for each $x\in R_i$ we have

(14) $$d_{R_0}(x)\ge(1/3-\epsilon)|R_0|.$$

Again this is possible by Theorem 3(iii). We also let $B_i=X_i\setminus R_i$. Then note that, in particular, for each $y\in R_0$ and at least $(1-2\epsilon)k_1$ different values of $i\in[k_1]$ we have

(15) $$d_{R_i}(y)\ge(1/3-\epsilon)|R_i|-\epsilon\tilde n>5\epsilon\tilde n.$$

This completes the preparation of H_α.

Next we describe the general embedding procedure. We assume that we have already embedded subtrees $T_1,\dots,T_{c-1}$ into H_α. We let Q^c denote the set of roots $r(T_m)$ of subtrees T_m such that $r(T_m)$ is a vertex of $T_1\cup\ldots\cup T_{c-1}$. Then we also assume that we have *assigned* an index $\beta(m)$ for each m with $r(T_m)\in Q^c$, so if $1<c<q$ the number of indices we have assigned is greater than the number of subtrees we have embedded. If $c=1$ then we let $\beta(1)=\gamma(1)$. In particular, we have assigned the index $\beta(c)$ to c. We think of the statement $\beta(m)=j$ as meaning "space for T_m has been reserved in the pair (B_j,Y_j)", and when it comes time to embed the subtree T_m we shall embed it in $(X_{\beta(m)},Y_{\beta(m)})$. Below for each ℓ, $1\le\ell\le c-1$, we denote by ϕ^ℓ the embedding of $T_1\cup\ldots\cup T_{\ell-1}$. For a subset S of vertices we write S^ℓ for $S\setminus\phi^\ell(T_1\cup\ldots\cup T_{\ell-1})$, the subset of S which has not yet been used by ϕ^ℓ. As above, we let Q^ℓ denote the set of $r(T_m)$ such that $r(T_m)\in V(T_1\cup\ldots\cup T_{\ell-1})$, and let $q^{(\ell)}=|Q^\ell|$ (so $c\le q^{(\ell)}\le q$).

We assume that for each ℓ, $1\le\ell\le c$, the embedding ϕ^ℓ of $T_1\cup\ldots\cup T_{\ell-1}$ has the following properties.

(A) Each root in Q^c is embedded into a vertex of $R_0\cup\bigcup_{i=1}^k R_i$. In particular, the root $r(T_c)$ of T_c is embedded into a vertex $r_c\in R_0\cup\bigcup_{i=1}^k R_i$. (If $c=1$ we just embed $r(T_1)$ into any vertex of R_0 if $r(T_1)\in V_2(T)$, or into any vertex of R_1 if $r(T_1)\in V_1(T)$.)

(B) For each m such that $r(T_m)\in Q^c\cap V_2(T)$ we have

$$d_{R_{\beta(m)}}(\phi^c(r(T_m)))>5\epsilon\tilde n\,.$$

In particular, $d_{R_{\beta(c)}}(r_c)>5\epsilon\tilde n$.

(C) We have

$$|R_0^c|\ge|R_0|-(c-1)\delta t_2-q^{(c)}.$$

We find from (1), (2), (3), and the definition of δ that

$$\begin{aligned}(c-1)\delta t_2-q^{(c)}&<q(\delta t_2+1)<2q\delta t_2<2t_2\frac{12k}{\epsilon}\frac{\epsilon^2}{100k_1^2}\le\frac{24\epsilon t_2}{100k}\\&<\frac{24\epsilon n\alpha}{100k(\alpha+2)}\le\frac{24\epsilon\alpha\tilde n}{100(1-\epsilon)}<\frac{96\epsilon\tilde n}{100}<\epsilon\tilde n\,,\end{aligned}\tag{16}$$

which implies that

$$|R_0^c|\ge|R_0|-\epsilon\tilde n.\tag{17}$$

(D) For each i, $1\le i\le k$, we have

$$|R_i^c|\ge|R_i|-(c-1)\delta t_2-q^{(c)}$$

where from (16) we find

$$|R_i^c| \geq |R_i| - \epsilon\tilde{n} \geq 20\epsilon\tilde{n} \tag{18}$$

for each i, $1 \leq i \leq k$.

(E) For $\ell \leq c-1$, the subtree T_ℓ is embedded such that $V(T_\ell) \setminus Q^{\ell+1}$ is embedded into $X_j \cup Y_j \cup R_0$, where $j = \beta(\ell)$ is the index assigned to ℓ. Also

- all neighbours of $r(T_\ell)$ in $V(T_\ell) \setminus Q^{\ell+1}$ are embedded into R_0^ℓ if $r(T_\ell) \in V_1(T_\ell)$, and into R_j^ℓ if $r(T_\ell) \in V_2(T_\ell)$,
- the remaining vertices of $V_1(T_\ell) \setminus Q^{\ell+1}$ are embedded into B_j^ℓ, and the remainder of $V_2(T_\ell) \setminus Q^{\ell+1}$ into Y_j^ℓ.

(F) For all but at most $4\epsilon q^{(c)}$ values of m, where $r(T_m) \in Q^c$, the index $\beta(m)$ assigned to m is equal to $\gamma(m)$ (see Lemma 5). If this happens we call T_m *normal*, otherwise it is *exceptional*. If T_m is exceptional then $\beta(m) \in [k_1]$. We let $E^c \subset Q^c$ denote the set of $r(T_m) \in Q^c$ for which T_m is exceptional.

(G) For each i, the amount of space reserved in B_i is less than $|B_i| - 79\epsilon\tilde{n}$ and the amount reserved in Y_i is less than $|Y_i| - 79\epsilon\tilde{n}$. More formally, we have

$$\left|\bigcup\{V_1(T_m) : r(T_m) \in Q^c, \beta(m) = i\}\right| < |B_i| - 79\epsilon\tilde{n},$$

and

$$\left|\bigcup\{V_2(T_m) : r(T_m) \in Q^c, \beta(m) = i\}\right| < |Y_i| - 79\epsilon\tilde{n}.$$

In particular, this implies that $|B_i^c| > 79\epsilon\tilde{n}$ and $|Y_i^c| > 79\epsilon\tilde{n}$.

To show that T_c can be embedded into the available part of $(X_{\beta(c)}, Y_{\beta(c)})$, we first need the following.

Lemma 6. *Let $j = \beta(c)$ (see (E)). Then there exist subsets $\bar{B}_j \subset B_j^c$, $\bar{Y}_j \subset Y_j^c$, $\bar{R}_j \subset R_j^c$, and $\bar{R}_0 \subset R_0^c$ such that*

(i) *for $y \in \bar{R}_0 \cup \bar{Y}_j$ we have $d_{\bar{B}_j}(y) \geq \epsilon\tilde{n}$ and $d_{\bar{R}_j}(y) \geq \epsilon\tilde{n}$,*
(ii) *for $x \in \bar{R}_j \cup \bar{B}_j$ we have $d_{\bar{R}_0}(x) \geq (1/3 - 7\epsilon)|R_0| \geq \epsilon\alpha\tilde{n}$ and $d_{\bar{Y}_j}(x) \geq \epsilon\alpha\tilde{n}$,*
(iii) *$|\bar{R}_0| \geq |R_0| - 3\epsilon\alpha\tilde{n} \geq (1/2 - 3\epsilon)\alpha\tilde{n}$,*
(iv) *if $r_c \in R_0$ we have $d_{\bar{R}_j}(r_c) \geq \epsilon\tilde{n}$.*

Proof. We begin by defining the sets $\bar{B}_j$, $\bar{Y}_j$, $\bar{R}_j$, and $\bar{R}_0$. By Theorem 3(iii) we know there exists a set $\bar{B}_j \subset B_j^c$ with

$$|\bar{B}_j| \geq |B_j^c| - 2\epsilon\tilde{n} \geq 6\epsilon\tilde{n}, \tag{19}$$

where we use (G), such that for each $x \in \bar{B}_j$ we have

$$d_{R_0^c}(x) \geq (1/3 - \epsilon)|R_0^c| \tag{20}$$

and

$$d_{Y_j^c}(x) \geq (1/3 - \epsilon)|Y_j^c|. \tag{21}$$

Also, again using (G), there exists a set $\bar{Y}_j \subset Y_j^c$ with

$$|\bar{Y}_j| \geq |Y_j^c| - 2\epsilon\alpha\tilde{n} \geq 10\epsilon\alpha\tilde{n} \tag{22}$$

such that for each $y \in \bar{Y}_j$ we have

$$d_{\bar{B}_j}(y) \geq (1/3 - \epsilon)|\bar{B}_j| \geq \epsilon\tilde{n}, \tag{23}$$

where the last inequality follows from (19), and

$$d_{R_j^c}(y) \geq (1/3 - \epsilon)|R_j^c|. \tag{24}$$

Now there exists a set $\bar{R}_j \subset R_j^c$ with

$$|\bar{R}_j| \geq |R_j^c| - \epsilon\tilde{n} \geq 19\epsilon\tilde{n}, \tag{25}$$

where here we use (18), such that for each $x \in \bar{R}_j$ we have

$$d_{\bar{Y}_j}(x) \geq (1/3 - \epsilon)|\bar{Y}_j| \geq \epsilon\alpha\tilde{n}, \tag{26}$$

where the last inequality follows from (22). Finally there exists a set $\bar{R}_0 \subset R_0^c$ with

$$|\bar{R}_0| \geq |R_0^c| - 2\epsilon\alpha\tilde{n} \geq |R_0| - 3\epsilon\alpha\tilde{n} \geq 17\epsilon\alpha\tilde{n}, \tag{27}$$

where we use (17), such that for each $y \in \bar{R}_0$ we have

$$d_{\bar{B}_j}(y) \geq (1/3 - \epsilon)|\bar{B}_j| \geq \epsilon\tilde{n}, \tag{28}$$

where the last inequality follows from (19), and

$$d_{\bar{R}_j}(y) \geq (1/3 - \epsilon)|\bar{R}_j| \geq \epsilon\tilde{n}, \tag{29}$$

where we use (27).

Now we prove the properties (i) to (iv). First we consider (i). Note that for $y \in \bar{R}_0 \cup \bar{Y}_j$, the fact that $d_{\bar{B}_j}(y) \geq \epsilon\tilde{n}$ is given by (23) and (28). Also, if $y \in \bar{Y}_j$, then by (18), (24) and (25), we find

$$d_{\bar{R}_j}(y) \geq d_{R_j^c}(y) - (|R_j^c| - |\bar{R}_j|) \geq (1/3 - \epsilon)|R_j^c| - \epsilon\alpha\tilde{n} \geq \epsilon\tilde{n}.$$

Therefore this together with (29) proves (i).

Next we turn to (ii). For $x \in \bar{R}_j$, we have by (14), (27), and the definition of R_0 that

$$d_{\bar{R}_0}(x) \geq d_{R_0}(x) - (|R_0| - |\bar{R}_0|)$$

$$\geq (1/3-\epsilon)|R_0| - 3\epsilon\alpha\tilde{n} \geq (1/3-7\epsilon)|R_0| \geq \epsilon\alpha\tilde{n}.$$

This together with (26) proves the assertion for $x \in \bar{R}_j$.

For $x \in \bar{B}_j$ we have by (17), (20), and (27)

$$\begin{aligned} d_{\bar{R}_0}(x) &\geq d_{R_0^c}(x) - (|R_0^c| - |\bar{R}_0|) \\ &\geq (1/3-\epsilon)|R_0^c| - 2\epsilon\alpha\tilde{n} \geq (1/3-7\epsilon)|R_0| \geq \epsilon\alpha\tilde{n}, \end{aligned}$$

and by (21) and (22) and (G) we find

$$d_{\bar{Y}_j}(x) \geq d_{Y_j^c}(x) - (|Y_j^c| - |\bar{Y}_j|) \geq (1/3-\epsilon)|Y_j^c| - 2\epsilon\alpha\tilde{n} \geq \epsilon\alpha\tilde{n}.$$

This completes the proof of (ii).

Part (iii) is immediate from (27). To check (iv), note that

$$d_{\bar{R}_j}(r_c) > d_{R_j}(r_c) - (|R_j| - |R_j^c|) - (|R_j^c| - |\bar{R}_j|) > \epsilon\tilde{n}$$

by (B), (18), and (25). ∎

To complete the proof, we describe how to extend the embedding ϕ^ℓ and the index assignment β so that (A) to (G) are still satisfied.

Lemma 7. *Any embedding ϕ^c and index assignment β which satisfy Properties (A) to (G) can be extended to an embedding ϕ^{c+1} of $T_1 \cup \ldots \cup T_c$ and an assignment β of $\{m : r(T_m) \in Q^{c+1} \setminus Q^c\}$ for which Properties (A) to (G) hold as well.*

Proof. We consider two cases according to whether $r_c \in R_0$ or $r_c \in \bigcup_{i=1}^k R_i$ (see (A)). Let $j = \beta(c)$.

First suppose $r_c \in R_0$. Our basic approach will be to use a greedy embedding to place into $\bar{B}_j \cup \bar{R}_j \cup \bar{Y}_j$ the vertices of T_c that are not in the set $S = (Q^{c+1} \setminus Q^c) \cap V_2(T)$, one vertex at a time. When we encounter a vertex v of S, we shall define $\beta(\ell)$ for all $\ell \in L = \{\ell : c < \ell \leq q, r(T_\ell) = v\}$, and we must take special care to embed v into R_0 such that (B) and (G) are satisfied for each $\ell \in L$, and $\beta(\ell) = \gamma(\ell)$ for at least $(1-4\epsilon)|L|$ elements of L (see (F) and Lemma 5).

We begin by embedding the neighbours of $r(T_c)$ in T_c into $\bar{R}_j$. Recall that there are at most $\delta t_2 < \epsilon\tilde{n}$ such neighbours, so this is possible by Lemma 6(iv). Then we complete the embedding one vertex at a time according to the following rules.

At each step we embed a vertex v of T_c which is adjacent in T_c to some vertex w we have already embedded. If v is not a root of any subtree T_m, we place v into $\bar{B}_j$ if $v \in V_1(T)$, and into $\bar{Y}_j$ if $v \in V_2(T)$. If v is a root of some subtree T_m and $v \in V_1(T)$ (so v is not in S), we place v into $\bar{R}_j$, and we let $\beta(m)=\gamma(m)$. Lemma 6 guarantees that such a greedy embedding is possible, since T_c has at most $s \le \epsilon\tilde{n}$ vertices.

If $v \in S$ then w has been embedded into a vertex $x \in \bar{B}_j \cup \bar{R}_j$. Let L be as defined above. By Lemma 6(ii) we know that $d_{\bar{R}_0}(x) \ge (1/3-7\epsilon)|R_0| > (1/3-7\epsilon)|\bar{R}_0|$. We want to choose a neighbour z of x in $\bar{R}_0$ and an index $\beta(\ell)$ for each $\ell \in L$ such that if we embed v into z then the required properties will be satisfied.

Now by Lemma 6(iii), for each $\ell \in L$ there are at most $\epsilon\alpha\tilde{n}$ vertices y of $\bar{R}_0$ such that $d_{R_{\gamma(\ell)}}(y) < (1/3-\epsilon)|R_{\gamma(\ell)}|$. Proceeding with a similar argument to the one which led to (15), we find that the number of vertices y in $\bar{R}_0$ such that $d_{R_{\gamma(\ell)}}(y) < (1/3-\epsilon)|R_{\gamma(\ell)}|$ for at least $4\epsilon|L|$ different values of $\ell \in L$ is at most $|\bar{R}_0|/4$. Therefore since $d_{\bar{R}_0}(x) \ge (1/3-7\epsilon)|\bar{R}_0|$, there exists a vertex $z \in \Gamma(x) \cap \bar{R}_0$ such that $d_{R_{\gamma(\ell)}}(z) \ge (1/3-\epsilon)|R_{\gamma(\ell)}|$ for at least $(1-4\epsilon)|L|$ different values of $\ell \in L$. We choose this z to embed the vertex v. Recall also by (15) that there exists a subset $K \subset [k_1]$ of size at least $(1-2\epsilon)k_1$ such that $d_{R_i}(z) > 5\epsilon\tilde{n}$ for all $i \in K$.

Finally we describe how the assignment β will be extended to L. If $\ell \in L$ is such that $d_{R_{\gamma(\ell)}}(z) \ge (1/3-\epsilon)|R_{\gamma(\ell)}|$, then we let $\beta(\ell)=\gamma(\ell)$, and the corresponding tree T_ℓ will be normal. Then note that (B) is satisfied for this value of ℓ, since $(1/3-\epsilon)|R_{\gamma(\ell)}| > 5\epsilon\tilde{n}$. If ℓ is one of the at most $4\epsilon|L|$ elements of L for which $d_{R_{\gamma(\ell)}}(z) < (1/3-\epsilon)|R_{\gamma(\ell)}|$, the tree T_ℓ will be exceptional, and we denote the set of such ℓ by $E(L)$, then we will have $E(L) \subset E^{c+1} \setminus E^c$. We shall let $\beta(\ell)$ be a carefully chosen element of K, so next we explain the procedure by which such elements are chosen. Note that choosing $\beta(\ell) \in K$ ensures that (B) will hold for each $\ell \in E(L)$.

Let $E(L)=\{\ell_1,\dots,\ell_t\}$, and suppose $\beta(\ell_i)$ has been chosen for $1 \le i \le b$. By (F), we know that the only subtrees T_m for which $\beta(m)=i$ for $i \in K$ are the exceptional subtrees T_m with $r(T_m) \in E^c$, together with $\{T_{\ell_i} : 1 \le i \le b\}$. Recall from (F) that $|E^c| \le 4\epsilon q^{(c)}$, and also we know $b < t \le 4\epsilon|L|$. Therefore since each subtree has at most $\epsilon\tilde{n}$ vertices, we find that the total number of vertices in $\bigcup_{i\in K} B_i$ that have been reserved so far is at most

$$(4\epsilon q^{(c)} + b)\epsilon\tilde{n} < 4\epsilon^2 q\tilde{n} < 48\epsilon k\tilde{n},$$

where we use (3). This implies that the number f of values of $i \in K$ for which more than $|B_i|$–$80\epsilon\tilde{n}$ space is reserved in B_i satisfies $(|B_i|–80\epsilon\tilde{n})f < 48\epsilon k\tilde{n}$, so since $|B_i| \ge (1–21\epsilon)\tilde{n}$ for each i and $\epsilon < 10^{-6}$, we find $f < (48\tilde{n}/(1–101\epsilon)\tilde{n})\epsilon k <$

$49\epsilon k$. Therefore there are at least

$$|K| - 49\epsilon k > (1-2\epsilon)100\epsilon k - 49\epsilon k > 50\epsilon k > |K|/2$$

values of $i \in K$ such that B_i has at least $80\epsilon\tilde{n}$ unreserved vertices. Similarly there are more than $50\epsilon k > |K|/2$ values of $i \in K$ for which Y_i has at least $80\epsilon\tilde{n}$ unreserved vertices. Therefore there exists $j \in K$ such that there are at least $80\epsilon\tilde{n}$ unreserved vertices in each of B_j and Y_j. We set $\beta(\ell_{b+1}) = j$. Doing this for each element of $E(L)$ completes the assignment on L. This then completes the extension of ϕ^c and β.

For the second case let us suppose that $r_c \in R_i$ for some i. Then by (14) and Lemma 6(iii) we find that

$$d_{\bar{R}_0}(r_c) \geq (1/3 - \epsilon)|R_0| - 3\epsilon\alpha\tilde{n} \geq \epsilon\alpha\tilde{n}\,.$$

Therefore the neighbours of r_c in T_c can be embedded into $\bar{R}_0$. We then embed the rest of T_c into $\bar{B}_j \cup \bar{Y}_j \cup \bar{R}_j \cup \bar{R}_0$, and extend the assignment β, following the same rules as in the previous case.

Finally we check that Properties (A) through (G) are satisfied with the above definitions. Properties (A) and (E) hold by construction, and (F) follows from our rules for embedding roots in $Q^{c+1} \backslash Q^c$. We noted already in the above description that (B) holds. To see (C), note that the only vertices that are embedded into R_0 in this construction are roots in $Q^{c+1} \setminus Q^c$, of which there are $q^{(c+1)} - q^{(c)}$, or (if $r_c \in \bigcup_{i=1}^k R_i$) neighbours of $r(T_c)$ in T_c, of which there are at most δt_2. Therefore (C) is satisfied. Property (D) holds for a similar reason. Finally, for $i \in \{k_1+1, \ldots, k\}$ we have that (G) follows from Lemma 5, since we have reserved space in (B_i, Y_i) only for those trees T_m for which $\beta(m) = \gamma(m)$. For $i \in [k_1]$, Property (G) holds by our choice of $\beta(\ell)$ for $\ell \in L$. ∎

Now Theorem 1 follows from Lemmas 6 and 7. ∎

6. Node restricted weight functions

In this section we introduce a special class of functions defined on arcs of a digraph, which we shall use in Section 8 to replace Szemerédi's partition of the two-coloured complete graph by another "non-balanced" partition in which sets are of different sizes.

Let α, $1 < \alpha \leq 2$, be a fixed real number (later we set $\alpha = |V_2|/|V_1|$, where (V_1, V_2) is the bipartition of the tree T we are to embed in the two-coloured complete graph) and let $\vec{G} = (V, E)$ be a digraph. For a vertex $v \in V$, by

$D^-(v)$ we denote the set of all arcs of $\vec{G}$ which have heads v; similarly, $D^+(v)$ stands for the set of all arcs leaving the vertex v. A non-negative function $f: E \to \mathbb{R}$ is a *node restricted weight function*, or simply *weight function* on $\vec{G}$, if for all $x \in V(\vec{G})$ we have

$$\sum_{e\in D^-(x)} f(e) + \frac{1}{\alpha} \sum_{e\in D^+(x)} f(e) \leq 1\,. \tag{30}$$

The set of all weight functions defined on $\vec{G}=(V,E)$ is denoted by $\mathcal{F}=\mathcal{F}(\vec{G})$. For each $f\in\mathcal{F}$, $x\in V$, we put

$$w^-(x,f) = \sum_{e\in D^-(x)} f(e) \quad \text{and} \quad w^+(x,f) = \sum_{e\in D^+(x)} f(e)/\alpha.$$

We set also $w(x,f) = w^-(x,f) + w^+(x,f)$ and $w(f) = \sum_{e\in E} f(e)$. Thus, the condition (30) can be written as $w(x,f) \leq 1$. Finally, for $U \subseteq V$, let $w(U,f) = \sum_{x\in U} w(x,f)$.

In the following sections we try to maximize the value of $w(f)$ for some special class of digraphs. Note that one can view $\mathcal{F}$ as a subset of $\mathbb{R}^{|E|}$, which is non-empty (since $0\in\mathcal{F}$), bounded (since it is contained in $[0,\alpha]^{|E|}$), and closed. Hence $w(f)$, being a continuous function of the coordinates, defined on a compact subset of euclidean space, must attain its maximum in at least one point. Thus, let

$$w_{\max} = \max\{w(f) : f \in \mathcal{F}\}\,,$$

and

$$\mathcal{F}_{\max} = \mathcal{F}_{\max}(\vec{G}) = \{f \in \mathcal{F} : w(f) = w_{\max}\}\,.$$

Note that both $\mathcal{F}$ and $\mathcal{F}_{\max}$ are non-empty convex subsets of $\mathbb{R}^{|E|}$.

The main result of this section states that in every digraph $\vec{G}$ the family $\mathcal{F}_{\max}$ is related to a certain partition of its vertices.

Theorem 8. *Let $\vec{G}=(V,E)$ be a digraph and*

$$\begin{aligned} A &= \{a \in V : w(a,f) < 1 \text{ for some } f \in \mathcal{F}_{\max}\} \\ B &= \{b \in V : ba \in E \text{ for some } a \in A\} \\ C &= V \setminus (A \cup B)\,. \end{aligned}$$

Then the following hold.

(i) *$\vec{G}$ contains no arcs $xy \in E$ with $x \in A\cup C$ and $y \in A$. In particular, $A\cap B=\emptyset$.*

(ii) *For every* $b\in B$ *and* $f\in\mathcal{F}_{\max}$ *we have* $w^+(b,f)=1$. *In particular,* $w(B,f)=|B|$.
(iii) *If* $bx\in E$, $b\in B$ *and for some* $f\in\mathcal{F}_{\max}$ *we have* $f(bx)>0$, *then* $x\in A$.
(iv) *For all* $c\in C$ *and every* $f\in\mathcal{F}_{\max}$ *we have* $w(c,f)=1$. *Equivalently,* $w(C,f)=|C|$.
(v) $|B|\le w_{\max}/\alpha$.
(vi) $|C|\le(1+1/\alpha)w_{\max}-(1+\alpha)|B|$.
(vii) $|B|+|C|<(1+1/\alpha)w_{\max}$.
(viii) $|A|>|V|-(1+1/\alpha)w_{\max}$.

Proof. Note first that if $xy\in E$, $y\in A$, and $x\in V\setminus A$, then $x\in B$, and so there are no arcs coming from C to A. Now suppose that there is $a_1a_2\in E$ such that $a_1,a_2\in A$. Then, for $i=1,2$, there exists $f_i\in\mathcal{F}_{\max}$ such that $w(a_i,f_i)<1$. Let $f=(f_1+f_2)/2$. Since $\mathcal{F}_{\max}$ is convex, $f\in\mathcal{F}_{\max}$; furthermore, $\delta=1-\max\{w(a_1,f),w(a_2,f)\}>0$. But then we can increase the weight of the arc a_1a_2 by $\delta/2$, obtaining a new $f'\in\mathcal{F}$ with $w(f')>w(f)$, which contradicts the fact that $f\in\mathcal{F}_{\max}$. Thus, (i) holds.

In order to show (ii) assume that $w^+(b,f_1)<1$ for some $b\in B$ and $f_1\in\mathcal{F}_{\max}$. By the definition of B there exist $a\in A$ and $f_2\in\mathcal{F}_{\max}$ such that $ba\in E$ and $w(a,f_2)<1$. Let $f=(f_1+f_2)/2$. Then $f\in\mathcal{F}_{\max}$, $\max\{w(a,f),w^+(b,f)\}<1$, and, by the definition of A and B, $w(b,f)=1$. Let δ be a positive constant defined as

$$\delta=\min\{(1-w(a,f))/\alpha,(1-w^+(b,f))\}\le w^-(b,f)\,.$$

Construct a new $f'\in\mathcal{F}$ from f by subtracting δ from the weights of edges from $D^-(b)$ and adding $\alpha\delta$ to $f(ba)$. Then $w(f')>w(f)$, contradicting the choice of $f\in\mathcal{F}_{\max}$.

Now let $b\in B$, $bx\in E$, and $f_1(bx)>0$ for some $f_1\in\mathcal{F}_{\max}$. Then from the definition of A and B we infer that for some $a\in A$ and $f_2\in\mathcal{F}_{\max}$ we have $ba\in E$ and $w(a,f_2)<1$. Set $f=(f_1+f_2)/2$ and $\delta=\min\{f(bx),1-w(a,f)\}>0$. Build a new f' from $f\in\mathcal{F}_{\max}$ by setting $f'(ba)=f(ba)+\delta/2$ and $f'(bx)=f(bx)-\delta/2$. Then, clearly, $f'\in\mathcal{F}_{\max}$ and $w(x,f')<1$. Hence $x\in A$.

To see (iv) notice that from the definition of A, for all $x\in V\setminus A$ and all $f\in\mathcal{F}_{\max}$ we have $w(x,f)=1$.

Let $f\in\mathcal{F}_{\max}$. Note that (i) and (iii) imply that the set E' of all arcs $e\in E$ such that $f(e)>0$ can be divided into three parts: the set E_1, which consists of all arcs which go from B to A, the set E_2 which contains all arcs joining A to C, and E_3 of the arcs with both ends in C. Observe also that if an arc e belongs to E_1, then its weight contributes $f(e)/\alpha$ to $w(B,f)$ and nothing to $w(C)$, if $e\in E_2$, then it increases $w(C,f)$ by $f(e)$ and leaves $w(B,f)$ unchanged, and, finally, each e from E_3 adds $(1+1/\alpha)f(e)$ to $w(C,f)$ and does not affect the value of $w(B,f)$. Thus, each arc e from E', either adds $f(e)/\alpha$ to $w(B,f)$ and nothing to $w(C,f)$, or adds at most $(1+1/\alpha)f(e)$ to $w(C,f)$ and nothing to $w(B,f)$. Consequently,

$$\alpha w(B,f) + \frac{\alpha}{1+\alpha} w(C,f) \le \sum_{e\in E'} f(e) = w_{\max}\,,$$

and, since by (ii) and (iv) we have $w(B,f)=|B|$ and $w(C,f)=|C|$,

$$\alpha(1+\alpha)|B| + \alpha|C| \le (1+\alpha)w_{\max}\,.$$

The above inequality immediately gives (v) and (vi), and (vii) is a straightforward consequence of (vi) and the fact that $\alpha \ge 1$. Note also that

$$\alpha|B\cup C| = \alpha(|B|+|C|) \le \alpha(1+\alpha)|B| + \alpha|C| \le (1+\alpha)w_{\max}\,.$$

Hence, since $A\cup B\cup C$ is a partition of V,

$$|A| \ge |V| - |B\cup C| \ge |V| - (1+1/\alpha)w_{\max}\,,$$

which completes the proof of Theorem 8. ∎

Let $\mathrm{supp}(f)=\{e\in E : f(e)\neq 0\}$. We show that there exists $f\in\mathcal{F}_{\max}$ for which $|\mathrm{supp}(f)|$ is not too large.

Lemma 9. *For every $\vec{G}=(V,E)$ there exists $f\in\mathcal{F}_{\max}$ such that $|\mathrm{supp}(f)|\le 2|V|-2$.*

Proof. Let $\vec{G}(V,E)$ be a digraph and let f be a function from $\mathcal{F}_{\max}$ for which $|\mathrm{supp}(f)|$ is minimized. Consider the auxiliary bipartite graph $H(\vec{G})$ obtained from $\vec{G}$ by replacing each vertex x by two vertices x' and x'', and each arc xy from $\mathrm{supp}(f)$ by an edge joining x' and y'' in $H(\vec{G})$. Suppose that $H(\vec{G})$ contains a cycle C, and let $\delta = f(e_0) > 0$ denote the smallest weight among all the edges which belong to C. Since C is even, all the edges of C can be split into two matchings. Define a new function f' on $\mathrm{supp}(f)$ by subtracting δ from the weights of each edge from the matching containing e_0 and adding δ to the weights of the edges from the other matching. Note that for every $x\in V$ we have $\sum_{e\in D^+(x)} f'(e)=\sum_{e\in D^+(x)} f(e)$ and $\sum_{e\in D^-(x)} f'(e)=\sum_{e\in D^-(x)} f(e)$. Hence $f'\in\mathcal{F}_{\max}$, but $|\mathrm{supp}(f')|<|\mathrm{supp}(f)|$, contradicting the choice of f. Thus, $H(\vec{G})$ contains no cycles, and, since the graph $H(\vec{G})$ has $2|V|$ vertices, $|\mathrm{supp}(f)|\le 2|V|-2$. ∎

7. Weight functions and graph colourings

In this section we use weight functions to study the structure of a two-coloured graph which is "almost complete", i.e., is obtained from the complete graph by deleting only few edges.

Let α, $1<\alpha\leq 2$ be a given constant and let $G=(V,E)$ denote a graph whose edges were coloured by two colours, red and blue. For $x\in V$, let $\vec{G}_r(x)$ $[\vec{G}_b(x)]$ denote the digraph with vertex set $V(\vec{G}_r(x))=V\setminus\{x\}$ such that yz is an arc of $\vec{G}_r(x)$ if and only if both pairs $\{x,y\}$ and $\{y,z\}$ are edges of G coloured red [blue]. Our main result on weight functions on $\vec{G}_r(x)$ and $\vec{G}_b(x)$ can be stated as follows.

Theorem 10. *For every α, $1<\alpha\leq 2$ there exists m_0, such that for every $m\geq m_0$ the following holds. Let $G=(V,E)$ be a graph on $(2+\alpha)m+11m_1$ vertices, where $3\leq m_1\leq m/500$, such that $|E|\geq\binom{|V|}{2}-m_1^2$, whose edges are coloured with two colours, red and blue. Then there exist a vertex $x\in V$, a colour $c\in\{r,b\}$, and a weight function f on $\vec{G}_c(x)$, such that $w(f)\geq\alpha m$ and $|\text{supp}(f)|<2|V|$.*

The proof of Theorem 10 is long and technical, so we split it into series of lemmas. In all of them we shall assume that G, m, and m_1 fulfill the assumption of Theorem 10.

Lemma 11. *Let us suppose that there are disjoint subsets X and Y of V such that all edges in G between X and Y are coloured with the same colour, and $|X|,|Y|\geq m+3m_1+1$ while $|X|+|Y|=(1+\alpha)m+6m_1+2$. Then there exist $x\in V$, a colour $c\in\{r,b\}$, and a weight function f_x on $\vec{G}_c(x)$, such that $w(f_x)\geq\alpha m$.*

Proof. Let us suppose that all edges between X and Y are coloured blue. Furthermore, we may and shall assume that $|X|=m+p+3m_1+1$ and $|Y|=\alpha m-p+3m_1+1$, for some p, $0\leq p\leq(\alpha-1)m$. Note also, that Y must contain a vertex x which is adjacent to at least $|X|-m_1$ vertices in X, since otherwise at least $|Y|m_1\geq m_1^2$ are missing from G, contradicting our assumptions. Similarly, some vertex $y\in X$ has at least $|Y|-m_1$ neighbours in Y.

Thus, let $x,y\in V$, $X_1,Y_1\subseteq V$ be chosen in such a way that:

- $|X_1|=m+p+2m_1$,
- $|Y_1|=\alpha m-p+2m_1$,
- all pairs $\{x,x_1\}$, where $x_1\in X_1$, are edges of G and are coloured blue,
- all pairs $\{y,y_1\}$, $y_1\in Y_1$, are edges of G coloured blue,

- all edges of G between the sets X_1 and Y_1 are coloured blue.

In order to simplify slightly our argument we delete from G all blue edges joining x with vertices outside X_1 and all blue edges which join y to vertices outside Y_1.

Let A_x, B_x, C_x and A_y, B_y, C_y denote the partitions defined by Theorem 8 in the digraphs $\vec{G}_b(x)$ and $\vec{G}_b(y)$ respectively. Our first aim will be to verify the following claim.

Claim 1. *Let the sets X_1, Y_1, A_x, B_x, C_x, A_y, B_y, C_y be defined as above. Then, either in one of the digraphs $\vec{G}_b(x)$ or $\vec{G}_b(y)$ there exists a weight function f with $w(f) \geq \alpha m$, or the following hold:*

(i) $|C_x \cap Y_1| > \alpha m - p + m_1$,
(ii) $|C_y \cap X_1| > m + p + m_1$,
(iii) $(1+\alpha)|B_x| + |C_x \setminus Y_1| < (1+1/\alpha)p$,
(iv) $(1+\alpha)|B_y| + |C_y \setminus X_1| < (1+1/\alpha)((\alpha-1)m - p)$,
(v) $|A_x \cap X_1| + |A_y \cap Y_1| + |(A_x \cup A_y) \setminus (X_1 \cup Y_1)| > 2m + m/\alpha + 10m_1$,
(vi) $|(C_x \cup C_y) \setminus (X_1 \cup Y_1)| < m$,
(vii) $|(A_x \cap A_y) \setminus (X_1 \cup Y_1)| > m_1$.

Proof of Claim 1. Let $\bar{f}$ denote the weight function which is the arithmetic mean of all weight functions from $\mathcal{F}_{\max}(\vec{G}_b(x))$. Then, for *every* $a \in A_x$, we have $w(a, \bar{f}) < 1$. We may assume that $w(\bar{f}) < \alpha m$ since otherwise we are done. Furthermore, Theorem 8(vii) implies that

$$|B_x| + |C_x| < (1 + 1/\alpha)w(\bar{f}) < (1 + \alpha)m\,.$$

Hence, since A_x, B_x and C_x form a partition of $X_1 \cup Y_1$, we have

$$|A_x \cap (X_1 \cup Y_1)| > 4m_1. \tag{31}$$

Note that, by Theorem 8(v), $|B_x| < w(\bar{f})/\alpha < m$ and, since $|X_1| > m + m_1$ and A_x, B_x and C_x form a partition of X_1,

$$|X_1 \cap (A_x \cup C_x)| > m_1. \tag{32}$$

Notice also that there are no arcs between $A_x \cup C_x$ and A_x in $\vec{G}_b(x)$ (Theorem 8(i)), and so there are no blue edges of G between $X_1 \cap (A_x \cup C_x)$ and $A_x \cap Y_1$. But, from our assumption, there are no red edges between these two sets as well. Hence, since in G at most m_1^2 edges are omitted, (32) implies that

$$|A_x \cap Y_1| < m_1. \tag{33}$$

Hence, because of (31), we have also

$$|A_x \cap X_1| > m_1. \tag{34}$$

Finally, let us recall that we have removed from G all its edges which join x to Y_1, and thus $B_x \cap Y_1 = \emptyset$. Hence, since $|Y_1| = \alpha m - p + 2m_1$, (a) follows from (33) and the fact that A_x, B_x and C_x form a partition of Y_1. Clearly, by symmetry, (b) can be verified by a similar argument.

Now we argue that

$$\sum_{c_1 \in C_x \cap X_1, c_2 \in C_x \cap Y_1} \bar{f}(c_1 c_2) \le m_1\,. \tag{35}$$

Indeed, suppose the above inequality does not hold. Then there are more than m_1 vertices in $C_x \cap Y_1$ which are heads of arcs $e \in \mathrm{supp}(\bar{f})$ which start at C_x. The inequality (34) and the fact that at most m_1^2 edges are missing from G imply that for some $a \in A_x \cap X_1$, $c_1 \in C_x \cap X_1$ and $c_2 \in C_x \cap Y_1$ we have $ac_2, c_1c_2 \in E(\vec{G}_b(x))$ and $\bar{f}(c_1c_2) > 0$. Set $\delta = \min\{1 - w(a, \bar{f}), \bar{f}(c_1c_2)\} > 0$. Then we can subtract δ from $\bar{f}(c_1c_2)$ and add it to $\bar{f}(ac_2)$ obtaining a new maximum weight function f of $\vec{G}_b(x)$ with $w(c_1, f) < 1$. This fact, however, contradicts Theorem 8(iv). Hence, (35) holds.

Let us split all arcs from $\mathrm{supp}(\bar{f})$ into three groups. The set E_1 contains all arcs coming from B_x to A_x, E_2 consists of all arcs joining A_x and C_x, and E_3 contains all arcs joining vertices of C_x. Note that, by Theorem 8(i) and (iii), $\mathrm{supp}(\bar{f}) = E_1 \cup E_2 \cup E_3$. Furthermore, each arc $e \in E_1$ contributes the term $\bar{f}(e)/\alpha$ to $w(B_x, \bar{f})$ and nothing to $w(C_x, \bar{f})$, each $e \in E_2$ adds $\bar{f}(e)$ to $w(C_x, \bar{f})$ and nothing to $w(B_x, \bar{f})$, and the only contribution of $e \in E_3$ is the term $(1 + 1/\alpha)\bar{f}(e)$ added to $w(C_x, \bar{f})$. Thus, by Theorem 8(ii),

$$|B_x| = w(B_x, \bar{f}) = \frac{1}{\alpha} \sum_{e \in E_1} \bar{f}(e)\,. \tag{36}$$

Similarly, Theorem 8(iv) gives

$$|C_x| = w(C_x, \bar{f}) = \sum_{e \in E_2} \bar{f}(e) + \frac{1+\alpha}{\alpha} \sum_{e \in E_3} \bar{f}(e)\,,$$

or, equivalently,

$$\frac{\alpha}{1+\alpha}|C_x| + \frac{1}{1+\alpha} \sum_{e \in E_2} \bar{f}(e) = \sum_{e \in E_2} \bar{f}(e) + \sum_{e \in E_3} \bar{f}(e)\,. \tag{37}$$

Since we may assume that $w(\bar f)<\alpha m$, from (36) and (37) we infer that

$$\alpha|B_x| + \frac{\alpha}{1+\alpha}|C_x| + \frac{1}{1+\alpha}\sum_{e\in E_2}\bar f(e) < \alpha m\,. \tag{38}$$

In order to estimate the sum in the above inequality, note first that from Theorem 8(iv) it follows that $w(C_x\cap Y_1,\bar f)=|C_x\cap Y_1|$. Furthermore, no arcs of $\vec G_b(x)$ are contained in Y_1, so all contributions to $w(C_x\cap Y_1,\bar f)$ come from arcs from E_2. Thus, from (35), we get

$$\sum_{e\in E_2}\bar f(e) \ge |C_x\cap Y_1| - m_1\,. \tag{39}$$

Combining (38) with (39), we arrive at

$$\begin{aligned} &\alpha|B_x| + \frac{\alpha}{1+\alpha}|C_x\cap Y_1| + \frac{\alpha}{1+\alpha}|C_x\setminus Y_1| + \frac{1}{1+\alpha}|C_x\cap Y_1| \\ &\quad = |C_x\cap Y_1| + \alpha|B_x| + \frac{\alpha}{1+\alpha}|C_x\setminus Y_1| < \alpha m + m_1\,. \end{aligned} \tag{40}$$

The above inequality, together with (a), gives (c). Similarly, by symmetry, one can prove (d).

Now note that the inequalities (c) and (d) imply that

$$|B_x| + |B_y| + |C_x\setminus Y_1| + |C_y\setminus X_1| < (\alpha - 1/\alpha)m\,. \tag{41}$$

Moreover, each vertex $v\in V\setminus\{x,y\}$ which does not belong to one of the three disjoint sets $A_x\cap X_1$, $A_y\cap Y_1$ and $A_x\cup A_y\setminus(X_1\cup Y_1)$ must be contained in at least one of the sets B_x, B_y, $C_x\setminus Y_1$, $C_y\setminus X_1$. Thus, (e) follows from (41) and the fact that $|V|=(2+\alpha)m+11m_1$.

In order to prove (f) observe that, by Theorem 8(iv) and the fact that x is not adjacent to any vertex outside X_1, for each $c\in C_x\setminus X_1$ we have $w^-(c,\bar f)=1$. Note also that for each arc $xc\in\mathrm{supp}(\bar f)$ we have $x\in X_1\cap(A_x\cup C_x)$; in particular, $x\notin(C_x\setminus X_1)\cup B_x$. Theorem 8(iii) implies that for each $bx\in\mathrm{supp}(\bar f)$ with $b\in B_x$ we have $x\in A_x$; and so again $x\notin(C_x\setminus X_1)\cup B_x$. Hence, each arc $e\in\mathrm{supp}(\bar f)$ has at most one end in $(C_x\setminus X_1)\cup B_x$. Moreover, if for an arc vw from $\mathrm{supp}\,\bar f$ we have $\{v,w\}\cap B_x\neq\emptyset$, then, by Theorem 8(ii), $v\in B_x$. Therefore, e contributes at most $f(e)$ to $\alpha w(B_x,\bar f)+w(C_x\setminus X_1,\bar f)$. Consequently, by Theorem 8(ii) and (iv),

$$\alpha|B_x| + |C_x\setminus X_1| = \alpha w(B_x,\bar f) + w(C_x\setminus X_1,\bar f) \le w(\bar f) < \alpha m\,,$$

i.e.,

$$\alpha|B_x| + |C_x \setminus (X_1 \cup Y_1)| + |C_x \cap Y_1| < \alpha m\,.$$

Thus, using (a), we infer that, in particular,

$$|C_x \setminus (X_1 \cup Y_1)| < \alpha m - |C_x \cap Y_1| < p\,.$$

Similarly,

$$|C_y \setminus (X_1 \cup Y_1)| < \alpha m - m - p\,.$$

Hence, combining the last two inequalities we get

$$|(C_x \cup C_y) \setminus (X_1 \cup Y_1)| < \alpha m - m \le m\,,$$

and (f) follows.

Finally, note that

$$|V \setminus (\{x,y\} \cup X_1 \cup Y_1)| = m + 7m_1 - 2 > m + m_1\,. \tag{42}$$

Since no vertex from $V\setminus(\{x,y\}\cup X_1\cup Y_1)$ is adjacent to either x or y in blue (we have removed all such edges from G), none of them belong to $B_x \cup B_y$. Thus, every vertex from $V\setminus(\{x,y\}\cup X_1\cup Y_1)$ is contained in exactly one of the two sets $(A_x\cap A_y)\setminus(X_1\cup Y_1)$ and $(C_x\cup C_y)\setminus(X_1\cup Y_1)$. Thus, (g) follows from (42) and (f).

This completes the proof of Claim 1. ∎

Theorem 8(i) implies that there are no arcs in $\vec{G}_b(x)$ which come from $(A_x \cup C_x)$ to A_x; consequently, no blue edges of G join $(A_x\cup C_x)\cap X_1$ and $A_x \setminus (X_1 \cup Y_1)$. Similarly, there are no blue edges between $(A_y \cup C_y)\cap Y_1$ and $A_y\setminus(X_1\cup Y_1)$. Thus, all edges of G joining the sets $(A_x\cup C_x)\cap X_1$ and $(A_y\cup C_y)\cap Y_1$ with $(A_x\cap A_y)\setminus(X_1\cup Y_1)$ are red. Hence, using (g) and the fact that there are at most m_1^2 edges missing from G, we infer that there exists a vertex $z\in(A_x\cap A_y)\setminus(X_1\cup Y_1)$ which is connected by red edges to all but at most m_1 vertices of the set $[(A_x\cup C_x)\cap X_1]\cup[(A_y\cup C_y)\cap Y_1]$.

Now let A_z, B_z and C_z be the partition of $\vec{G}_r(z)$ defined as in Theorem 8.

Claim 2. *If for none of the digraphs $\vec{G}_b(x)$, $\vec{G}_b(y)$, $\vec{G}_r(z)$ there exists a weight function f with $w(f)\ge\alpha m$, then the following hold:*

(a) $|A_x\cap A_z\cap X_1| < 2m_1$,
(b) $|A_y\cap A_z\cap X_1| < 2m_1$,
(c) $|(A_x\cap A_z)\setminus(X_1\cup Y_1)| < 2m_1$,
(d) $|(A_y\cap A_z)\setminus(X_1\cup Y_1)| < 2m_1$,
(e) $|B_z| < m/\alpha$,
(f) $|(A_x\cup C_x)\cap A_z\cap X_1| < m_1$,

(g) $|(A_y \cup C_y) \cap A_z \cap Y_1| < m_1$,
(h) $|[(A_x \cup A_y) \cap A_z] \setminus (X_1 \cup Y_1)| < m_1$.

Proof of Claim 2. Assume that $|A_x \cap A_z \cap X_1| \geq 2m_1$. Note that, since each vertex of X_1 is adjacent to x and, by Theorem 8(i), no arcs of $\vec{G}_b(x)$ are contained in A_x, no blue edges of G are contained in $A_z \cap A_x \cap X_1$. Theorem 8(i) implies also that there are no red edges $a_1 a_2$ in G such that $a_1, a_2 \in A_z \cap A_x \cap X_1$ and at least one of vertices a_1 and a_2 is adjacent to z. However, z was chosen in such a way that all but at most m_1 vertices of $A_x \cap X_1$ are adjacent to z in red. Hence, since $|A_z \cap A_x \cap X_1| \geq 2m_1$, at least m_1 of the vertices of $A_z \cap A_x \cap X_1$ are adjacent to z. But then, clearly, G is missing at least m_1^2 edges, contradicting the choice of G. Thus (a) holds. Clearly, (b) can be proved by a similar argument.

Now let us suppose that

$$|(A_x \cap A_z) \setminus (X_1 \cup Y_1)| \geq 2m_1 \,. \tag{43}$$

Using Theorem 8(i), one can argue as in the proof of (a) above that there are no blue edges in G between the sets $(A_x \cup C_x) \cap (A_z \cup C_z) \cap X_1$ and $(A_x \cap A_z) \setminus (X_1 \cup Y_1)$, and no vertex of $(A_x \cup C_x) \cap (A_z \cup C_z) \cap X_1$, which is connected to z by a red edge, has red neighbours in $(A_x \cap A_z) \setminus (X_1 \cup Y_1)$. However, by the choice of z, all but at most m_1 vertices from $(A_x \cup C_x) \cap (A_z \cup C_z) \cap X_1$ are adjacent to z in red. Thus, we must have

$$|(A_x \cup C_x) \cap (A_z \cup C_z) \cap X_1| < 2m_1 \tag{44}$$

since otherwise, more than m_1^2 edges would be missing from G. Note also that the sets A_x, B_x, C_x, as well as the sets A_z, B_z, C_z, form a partition of X_1. Thus, by (44),

$$\begin{aligned} |B_z \cap X_1| &\geq |(A_x \cup C_x) \cap X_1| - |(A_x \cup C_x) \cap (A_z \cup C_z) \cap X_1| \\ &> |(A_x \cup C_x) \cap X_1| - 2m_1 = |X_1| - |B_x| - 2m_1 \,. \end{aligned}$$

Since Claim 1(c) implies, in particular, that $|B_x| \leq p/\alpha$, it gives

$$|B_z \cap X_1| > m + p + 2m_1 - p/\alpha - 2m_1 \geq m \,.$$

Hence, by Theorem 8(v), there exists $f_z \in \mathcal{F}(\vec{G}_r(z))$ with $w(f_z) \geq \alpha m$ contradicting the assumption. Thus, (c) follows. The inequality (d) is, again, the symmetric equivalent of (c).

Recall that Claim 1(e) gives

$$|A_x \cap X_1| + |A_y \cap Y_1| + |(A_x \cup A_y) \setminus (X_1 \cup Y_1)| > 2m + m/\alpha + 10m_1 \,.$$

Furthermore, because of (a), (b), (c) and (d), fewer than $8m_1$ of the elements of A_z belong to the sets $A_x \cap X_1$, $A_y \cap Y_1$, $(A_x \cup A_y) \setminus (X_1 \cup Y_1)$. Thus, since A_z, B_z and C_z form a partition of $V \setminus \{x,y,z\}$,

$$|B_z| + |C_z| > 2m + m/\alpha. \tag{45}$$

However, Theorem 8(vi) implies that if there is no weight function f_z on $\vec{G}_r(z)$ with $w(f_z) \geq \alpha m$, then

$$(1+\alpha)|B_z| + |C_z| < (1+\alpha)m\,. \tag{46}$$

Subtracting (45) from (46) we arrive at

$$|B_z| < m\Big(1 - \frac{1}{\alpha} - \frac{1}{\alpha^2}\Big) < m\Big(1 - \frac{1}{\alpha}\Big) \leq \frac{m}{\alpha}\,, \tag{47}$$

and (e) follows.

Claim 1(c) implies, in particular, that

$$|B_x| + |C_x \cap X_1| < (1 + 1/\alpha)p\,.$$

The sets A_x, B_x and C_x form a partition of X_1, therefore

$$\begin{aligned} |A_x \cap X_1| &> |X_1| - (1 + 1/\alpha)p \\ &\geq m + 2m_1 - p/\alpha \geq m/\alpha + 2m_1\,, \end{aligned} \tag{48}$$

since $p \leq (\alpha - 1)m$. Similarly,

$$|A_y \cap Y_1| > m/\alpha + 2m_1\,,$$

and consequently, from (e),

$$\min\{|(A_x \cap X_1) \setminus B_z|, |(A_y \cap Y_1) \setminus B_z|\} > 2m_1\,. \tag{49}$$

Theorem 8(i) and the definition of $\vec{G}_b(x)$ imply that there are no blue edges of G between sets $A_x \cup C_x \supseteq (A_x \cup C_x) \cap A_z \cap X_1$ and $A_x \supseteq (A_x \cap C_x) \setminus B_z$. Similarly, one can infer that G contains no red edges vw such that $v \in (A_x \cup C_x) \cap A_z \cap X_1$, $w \in (A_x \cup C_x) \setminus B_z$, and w is a red neighbour of z. Note also that, by the choice of z, at most m_1 vertices of $A_x \cap X_1$ are not connected to z by red edges; hence, by (49), at least m_1 vertices from $(A_x \cap X_1) \setminus B_z$ are red neighbours of z. Consequently, $|(A_x \cup C_x) \cap A_z \cap X_1| < m_1$, since otherwise at least m_1^2 would be missing from G, and so (f) follows. Its counterpart (g) can be shown in a similar way.

Finally, notice that, arguing as before, one can infer that there is no edge e of G joining $(A_x \cap X_1) \setminus B_z$ and $(A_x \cap A_z) \setminus (X_1 \cup Y_1)$, or joining $(A_y \cap Y_1) \setminus B_z$ and $(A_y \cap A_z) \setminus (X_1 \cup Y_1)$, such that the end of e which belongs to either $(A_x \cap X_1) \setminus B_z$ or $(A_y \cap Y_1) \setminus B_z$ is a red neighbour of z. Furthermore, by (49), and the choice of z, at least m_1 vertices from each of the sets $(A_x \cap X_1) \setminus B_z$ and $(A_y \cap Y_1) \setminus B_z$ are adjacent to z. Therefore, (h) must hold, since otherwise G would be missing m_1^2 edges.

This completes the proof of Claim 2. ∎

In order to complete the proof of Lemma 11, note that, by Claim 2(f), (g) and (h), fewer than $5m_1$ elements from A_z belong to one of the sets $(A_x \cup C_x) \cap X_1$, $(A_y \cup C_y) \cap Y_1$, and $(A_x \cup A_y) \setminus (X_1 \cup Y_1)$. Furthermore, every vertex from $V \setminus \{x, y, z\}$ is either in one of the sets $(A_x \cup C_x) \cap X_1$, $(A_y \cup C_y) \cap Y_1$, $(A_x \cup A_y) \setminus (X_1 \cup Y_1)$, or in one of B_x, B_y, $(C_x \cap C_y) \setminus (X_1 \cup Y_1)$. Hence,

$$|A_z| < |B_x| + |B_y| + |(C_x \cap C_y) \setminus (X_1 \cup Y_1)| + 5m_1 \,. \tag{50}$$

Notice also that

$$\begin{aligned} |(C_x \cap C_y) \setminus (X_1 \cup Y_1)| &\le \min\{|C_x \setminus Y_1|, |C_y \setminus X_1|\} \\ \le \frac{1}{2}(|C_x \setminus Y_1| + |C_y \setminus X_1|) &\le \frac{\alpha}{1+\alpha}(|C_x \setminus Y_1| + |C_y \setminus X_1|) \,. \end{aligned}$$

Thus, using Claim 1(c) and (d), we get

$$\begin{aligned} &|B_x| + |B_y| + |(C_x \cap C_y) \setminus (X_1 \cup Y_1)| \\ &\le \frac{\alpha}{1+\alpha}[(1+\alpha)|B_x| + |C_x \setminus Y_1| + (1+\alpha)|B_y| + |C_y \setminus X_1|] \\ &< (\alpha - 1)m \le m \,. \end{aligned}$$

Hence, (50) gives $|A_z| < m + 5m_1$. But then

$$(1+\alpha)|B_z| + |C_z| > |B_z| + |C_z| > |V| - 3 - |A_z| > (1+\alpha)m \,,$$

and, by Theorem 8(vii), there exists a weight function f_z on $\vec{G}_r(z)$ such that $w(f,z) > \alpha m$. This completes the proof of Lemma 11. ∎

Lemma 12. *Let X and Y be two disjoint subsets of V, each containing $m+3m_1$ vertices, and let $v \in V \setminus (X \cup Y)$ be a vertex adjacent to all vertices from $X \cup Y$, such that all edges between v and $X \cup Y$ are coloured with the same colour, say, blue. If all edges of G between X and Y are also coloured blue, then there exists $z \in V$, a colour $c \in \{r, b\}$, and $f_z \in \mathcal{F}(\vec{G}_c(z))$, such that $w(f_z) \ge \alpha m$.*

Proof. Let us suppose that v, X and Y fulfill the assumptions of the lemma. Let f be a weight function in $\vec{G}_b(v)$ of maximum weight, and A, B, and C be set defined as in Theorem 8. Assume also that $w(f) < \alpha m$. Then, by Theorem 8(v),

$$|B| < m\,, \tag{51}$$

and so

$$\min\{|A \cup C) \cap X|, |(A \cup C) \cap Y|\} \geq 3m_1 > m_1\,. \tag{52}$$

Note that Theorem 8(i) implies that there are no blue arcs of $\vec{G}_b(v)$ between $(A \cup C) \cap Y$ and $A \cap X$, and thus, that there are no edges of G joining these two sets. Since there are at most m_1^2 edges missing from G, from (52) we get $|A \cap X| < m_1$. Similarly, $|A \cap Y| < m_1$. Hence

$$|A \cap (X \cup Y)| < 2m_1$$

and

$$|(B \cup C) \cap (X \cup Y)| > 2m + 4m_1\,. \tag{53}$$

Let $Z = V \setminus (\{v\} \cup X \cup Y)$. We shall show that the sets $(A \cup C) \cap (X \cup Y)$ and $A \cap Z$ fulfill the assumptions of Lemma 11 with colour red.

Note first that from (51) we get

$$\begin{aligned} |(A \cup C) \cap (X \cup Y)| &= |X \cup Y| - |B| \\ &> m + 6m_1 > m + 3m_1 + 1\,. \end{aligned} \tag{54}$$

From Theorem 8(vi) we infer that

$$|B| + |C| \leq (1 + \alpha)m - \alpha|B| \leq (1 + \alpha)m - |B|\,,$$

so, by (53),

$$\begin{aligned} |C \cap Z| < |(B \cup C) \cap Z| &= |B \cup C| - |(B \cup C) \cap (X \cup Y)| \\ &< (1 + \alpha)m - |B| - 2m - 4m_1 < (\alpha - 1)m - |B|\,. \end{aligned} \tag{55}$$

The sets B, $C \cap Z$, $(A \cup C) \cap (X \cup Y)$ and $A \cap Z$ form a partition of $V \setminus \{v\}$, so from (51) and (55) we get

$$\begin{aligned} |(A \cup C) \cap (X \cup Y)| + |A \cap Z| &> (2 + \alpha)m + 11m_1 - 1 - (\alpha - 1)m \\ &> 3m + 10m_1; \end{aligned} \tag{56}$$

in particular,

$$|(A \cup C) \cap (X \cup Y)| + |A \cap Z| > (1+\alpha)m + 6m_1 + 2. \tag{57}$$

Since $|X\cup Y| = 2m + 6m_1$, from (56) we get

$$|A \cap Z| > m + 4m_1 \geq m + 3m_1 + 1\,. \tag{58}$$

Finally, observe that $\vec{G}_b(v)$ contains no arcs coming from $(A\cup C)\cap(X\cup Y)$ to $A\cap Z$, and since all vertices of $X\cup Y$ are joined to v by blue edges of G, by Theorem 8(i), no blue edge of G joins $(A\cup C)\cap(X\cup Y)$ and $A\cap Z$. Thus, the assertion follows from Lemma 11, (54), (57) and (58). ∎

Lemma 13. *Let $v\in V$, $Q\subseteq V$ be the set of all vertices of G joined to v by blue edges, and $R = V\setminus(\{v\}\cup Q)$. Furthermore, let A, B, C be the partition of $\vec{G}_b(v)$ defined as in Theorem 8, $f\in\mathcal{F}_{\max}(\vec{G}_b(v))$, and $w_{\max} = w(f) < \alpha m$. Then the following hold:*

(i) $B\cap R = \emptyset$,
(ii) *all edges between the sets $Q\cap A$, $Q\cap C$ and $R\cap A$, and all edges inside $Q\cap A$, are red,*
(iii) $|Q\cap A| + |Q\cap C| + |R\cap A| > 2m + 10m_1$.

Proof. The definition of $\vec{G}_b(v)$, B and R immediately gives (i), while Theorem 8(i) implies (ii).

Now consider arcs $e\in \mathrm{supp}(f)$ which have at least one end in $(Q\cap B)\cup(R\cap C)$. Theorem 8(ii) states that for every $b\in B$ we have $w^+(b,f) = 1$, so e cannot end at $Q\cap B$. Furthermore, from the definition of $\vec{G}_b(v)$, no arc which belongs to this digraph starts at R. Thus, by Theorem 8(iii), either e starts at $Q\cap B$, or ends at $R\cap C$, but not both. Hence, such an arc e either adds $f(e)/\alpha$ to the value of $w(Q\cap B, f)$, or increases the value of $w(R\cap C)$ by $f(e)$. Thus, by Theorem 8(ii) and (iv),

$$\begin{aligned} |Q \cap B| + |R \cap C| &\leq \alpha|Q \cap B| + |R \cap C| \\ &= \alpha w(Q \cap B, f) + w(R \cap C, f) \leq w(f) \leq \alpha m. \end{aligned} \tag{59}$$

Now (iii) follows from (59) and the fact that, by (i), the sets $Q\cap A$, $Q\cap B$, $Q\cap C$, $R\cap A$ and $R\cap C$ form a partition of $x\in V\setminus\{v\}$. ∎

Lemma 14. *Let $x\in V$ be incident to at least $\alpha m+m_1$ edges of one colour, say blue, and let Q denote the set of all blue neighbours of v. Furthermore, let A, B, C be the partition of $V\setminus\{x\}$ defined as in Theorem 8, and let*

$$|Q\cap(A\cup C)|>m+5m_1\,. \tag{60}$$

Then, there exists $v\in V$, $c\in\{r,b\}$, and a weight function $f_v\in\mathcal{F}(\vec{G}_c(v))$, such that $w(f_v)\geq\alpha m$.

Proof. Let x and Q be defined as in the assumption of the lemma, and let $R=V\setminus(\{x\}\cup Q)$. Assume that for all $f\in\mathcal{F}(\vec{G}_b(x))$ we have $w(f)<\alpha m$. We consider the following two cases.

Case 1. $|Q\cap A|\geq m_1$.

Then, there exists a vertex $z\in Q\cap A$ which is adjacent to all but fewer than $2m_1$ vertices of $(Q\cap A)\cup(Q\cap C)\cup(R\cap A)$, since otherwise G would be missing m_1^2 edges. Furthermore, by Lemma 13(ii), all edges between z and the above set are red. Hence, from Lemma 13(iii), z has at least $2m+6m_1$ red neighbours in $(Q\cap A)\cup(Q\cap C)\cup(R\cap A)$. Moreover, by (60), z is incident to at least $m+3m_1$ vertices from $(Q\cap A)\cup(Q\cap C)$. Finally, by Theorem 8(viii),

$$|A|>|V|-1-(1+\alpha)m>m+10m_1, \tag{61}$$

and so z has at least

$$m+10m_1-2m_1>m+3m_1+1$$

red neighbours in A.

Now move some vertices of $Q\cap A$ from the set $X=(Q\cap A)\cup(Q\cap C)$ to $Y=R\cap A$ such that each of the resulting sets X', Y' has at least $m+3m_1$ elements. Then the vertex z and the sets X', Y', fulfill the assumption of Lemma 12 (in red) and the assertion follows.

Case 2. $|Q\cap A|<m_1$.

Theorem 8(vi) gives

$$\begin{aligned}|R\cap C|=|C|-|Q\cap C|&<(1+1/\alpha)m-(1+\alpha)|B|-|Q\cap C|\\&\leq(1+\alpha)m-2|B|-|Q\cap C|\,.\end{aligned}$$

Hence, using the fact that $|Q|\geq\alpha m+m_1$ and $|Q\cap A|<m_1$, we infer that

$$\begin{aligned}&|R\cap C|+|B|<(1+\alpha)m-|B|-|Q\cap C|\\&=(1+\alpha)m-|Q\cap(B\cup C)|=(1+\alpha)m-|Q|+|Q\cap A|<m.\end{aligned}$$

Since the sets B, $R\cap C$, $Q\cap A$, $Q\cap C$, $R\cap A$ form a partition of the set $V\setminus\{v\}$,

$$|Q \cap A| + |Q \cap C| + |R \cap A| > (1 + \alpha)m + 10m_1 \\ > (1 + \alpha)m + 6m_1 + 2\,. \tag{62}$$

Note now that $|Q\cap(A\cup C)|\geq m+3m_1+1$ by (60), and $|A|>m+3m_1+1$ by Theorem 8(viii) (cf. (61)). Thus, it is enough to adjust the sizes of the sets $Q\cap(A\cup C)$ and $R\cap A$ by moving some vertices from $Q\cap A$ to $R\cap A$ in such a way that the resulting sets have at least $m+3m_1+1$ vertices each, and apply Lemma 11. ∎

Lemma 15. *If G contains a vertex incident to $2m+5m_1$ edges of the same colour, then for some vertex $v\in V$, color $c\in\{r,b\}$, and weight function $f_v\in\mathcal{F}(\vec{G}_c(v))$ we have $w(f_v)\geq\alpha m$.*

Proof. Let $x\in X$ be a vertex with at least $2m+5m_1$ edges of the same colour, say blue, incident to it, and let Q denote the set of blue neighbours of x. Let A, B and C be the partition of $\vec{G}_b(x)$ defined as in Theorem 8 and $f\in\mathcal{F}_{\max}(\vec{G}_b(x))$. Suppose that $w(f)<\alpha m$. Then, from Theorem 8, we have $|B|<m$, and so

$$|Q\cap(A\cup C)| > |Q| - |Q\cap B| \geq |Q| - |B| > m + 5m_1\,.$$

Thus the assertion follows from Lemma 14. ∎

Proof of Theorem 10. Let G be a graph that fulfills the assumption of the Theorem. Then one can find a vertex $x\in G$ which is incident to at least $(1+\alpha/2)m+5m_1$ edges of one of the colours, say, blue. Let Q denote the set of all blue neighbours of x and let $R=V\setminus(\{x\}\cup Q)$. Thus,

$$|Q| \geq (1+\alpha/2)m + 5m_1\,. \tag{63}$$

Furthermore, let A, B and C be the partition of the vertex set of $\vec{G}_b(x)$ as defined in Theorem 8. As in the proof of Lemma 14 we consider the following two cases.

Case 1. $|Q\cap A|\geq m_1$.

Lemma 13(iii) states that the three sets $Q\cap A$, $Q\cap C$ and $R\cap A$ have at least $2m+10m_1$ vertices combined. Furthermore, since there are fewer than m_1^2 edges missing from G, and $|Q\cap A|\geq m_1$, there is a vertex $z\in Q\cap A$ which is adjacent to at least vertices $2m+m_1$ from $(Q\cap A)\cup(Q\cap C)\cup(R\cap A)$, and, by Lemma 13(ii), all edges joining z with this set are coloured red. Thus, from Lemma 15, we deduce that there exist $v \in V$, $c \in \{r,b\}$, and $f_v \in \vec{G}_c(v)$ with $w(f_v) \geq \alpha m$.

Case 2. $|Q \cap A| < m_1$.

Note first that (63) implies that

$$|Q \cap B| + |Q \cap C| = |Q| - |Q \cap A| > (1 + \alpha/2)m\,. \tag{64}$$

Furthermore, either there exists $f \in \mathcal{F}(\vec{G}_b(x))$ with $w(f) \geq \alpha m$, or, from Theorem 8(vi), we get

$$(1 + \alpha)|Q \cap B| + |Q \cap C| < (1 + 1/\alpha)m < (1 + \alpha m)\,. \tag{65}$$

Hence, combining (64) and (65), we infer that $|Q \cap B| > m/2$ and, consequently,

$$|Q \cap (A \cup C)| = |Q| - |Q \cap B| > (1 + \alpha/2)m + 5m_1 - m/2 \geq m + 5m_1.$$

Thus, Lemma 14 ensures that for some $v \in V$, $c \in \{r,b\}$ and $f_v \in \vec{G}_c(v)$ we have $w(f_v) \geq \alpha m$.

We have shown that for some colour $c \in \{r,b\}$ and vertex $v \in V$, for each weight function f defined on $\vec{G}_c(v)$ with the maximum weight we have $w(f) \geq \alpha m$. In order to complete the proof of Theorem 10 it is enough to observe that by Lemma 9 the function $f \in \mathcal{F}_{\max}(\vec{G}_c(v))$ can be chosen in such a way that $|\text{supp}(f)| \leq 2|V| - 2$. ∎

8. Proof of Theorem 3

In this section we use Theorem 10 to show Theorem 3. Thus, we shall show that if we colour edges of the complete graph with two colours and apply Szemerédi's Regularity Lemma to one of the colours, then one can use Theorem 10 to transform the partition obtained in this way into a non-balanced partition which contains a tree-like regular structure in one of the colours.

Proof of Theorem 3. Assume first that $\alpha \neq 1$, i.e., $1 < \alpha \leq 2$. Let $0 < \epsilon < 1/100$ and a two-colouring of the edges of K_n be given. Set $\epsilon_1 = \epsilon^5/4$ and let m_0 be such that the assertion of Theorem 10 holds. Apply the Regularity Lemma (Lemma 2) with ϵ_1 and $s_0 = m_0/\epsilon_1$ to find an (ϵ_1, s)-equitable partition $(U_0, U_1, \ldots, U_s)$ of the blue subgraph of K_n with $s \geq s_0$, where for every $i = 1, 2, \ldots, s$,

$$|U_i| = \hat{n} \geq (1 - \epsilon_1)n/s. \tag{66}$$

Consider an auxiliary graph G with vertex set $V(G)=\{1,2,\dots,s\}$ such that the pair $\{i,j\}$, $1\le i<j\le s$, is an edge of G if and only if the pair (U_i,U_j) is ϵ_1-regular in the blue graph (note that it is ϵ_1-regular in the red graph as well). Moreover, colour blue an edge $e=\{i,j\}$ of G if the density of (U_1,U_2) in the blue graph is at least $1/2$; otherwise colour e red. Then, the two-coloured G fulfills the assumptions of Theorem 10 for some m and m_1, where $m_1\le 20\epsilon_1 m$. Thus, there exist a vertex $i_0\in V(G)$, a colour, say, blue, and a weight function $f'\in\mathcal{F}(\vec{G}_b(i_0))$, such that

$$w(f')\ge \alpha m\ge (1-1000\epsilon_1)\frac{\alpha s}{2+\alpha}$$

and $|\operatorname{supp}(f)|\le 2s$. Consider a new weight function f obtained from f' by setting $f(e)=0$ on each arc e with $f'(e)\le 3\epsilon_1^2$. Then,

$$\begin{aligned} w(f)\ge w(f')-3\epsilon_1^2|\operatorname{supp}(f')| &\ge (1-1000\epsilon_1-18\epsilon^2)\frac{\alpha s}{2+\alpha}\\ &\ge (1-19\epsilon^2)\frac{\alpha s}{2+\alpha}\,. \end{aligned} \tag{67}$$

Now, for every arc $e=ij\in\operatorname{supp}(f)$, choose subsets $X^e\subseteq U_i$ and $Y^e\subseteq U_j$ such that $|X^e|=\lfloor\frac{f(e)}{\alpha}\hat{n}\rfloor$, $|Y^e|=\lfloor f(e)\hat{n}\rfloor$ and all subsets from the family $\{X^e,Y^e\colon e\in\operatorname{supp}(f)\}$ are disjoint (the inequality (30) used in the definition of a weight function guarantees that such a family exists). Finally, let Y_0 be any subset of U_{i_0} of $\lfloor\alpha\lfloor\epsilon^4\hat{n}\rfloor\rfloor$ elements.

Thus, we have constructed a family of sets $\{X^e,Y^e\colon e\in\operatorname{supp}(f)\}$ such that all sets are larger than $\epsilon^2\hat{n}$, and for every $e\in\operatorname{supp}(f)$ we have $|\alpha|X^e|-|Y^e||\le 2$. Now, let $\tilde{n}=\lfloor\epsilon^4\hat{n}\rfloor$. For every $e\in\operatorname{supp}(f)$ find in X^e a maximum family of disjoint subsets $X^e_1,X^e_2,\dots,X^e_{\ell(e)}$ of $\tilde{n}$ elements each, and, similarly, let $Y^e_1,Y^e_2,\dots,Y^e_{\ell(e)}$ be a disjoint family of subsets of Y^e, each of $\lfloor\alpha\tilde{n}\rfloor$ elements. Note that after this operation not more than $\epsilon^2|X^e|$ of the elements of X^e, and at most $\epsilon^2|Y^e|$ elements of Y^e, will not belong to one of the sets X^e_1, $X^e_2,\dots,X^e_{\ell(e)}$, Y^e_1, $Y^e_2,\dots,Y^e_{\ell(e)}$. Since $|U_{i_0}|\le\epsilon_1 n\le\epsilon^2 n$, from (66) and (67),

$$\begin{aligned} \sum_{e\in\operatorname{supp}(f)}\ \sum_{1\le i\le\ell(e)} &(|X^e_k|+|Y^e_k|)\\ &\ge (1-19\epsilon^2)\frac{\alpha s}{2+\alpha}\,\frac{1+\alpha}{\alpha}(1-\epsilon_1)\frac{n}{s}-2\epsilon^2 n\\ &> (1-25\epsilon^2)\frac{(1+\alpha)n}{2+\alpha}\,. \end{aligned}$$

Thus, the total number k of all pairs (X_i^e, Y_i^e), where $e \in \mathrm{supp}(f)$ and $i = 1, \ldots, \ell(e)$, is bounded from below by

$$k > (1-25\epsilon^2)\frac{(1+\alpha)n}{2+\alpha}\Big/(1+\alpha)\tilde{n} = \frac{1-25\epsilon^2}{2+\alpha}\frac{n}{\tilde{n}} > \frac{(1-\epsilon/2)n}{(2+\alpha)\tilde{n}},$$

and so

$$\tilde{n} > \frac{(1-\epsilon/2)n}{(2+\alpha)k}.$$

Observe also that for each $e \in \mathrm{supp}(f)$, $k = 1, 2, \ldots, \ell(e)$ we have $X_k^e \subseteq U_i$ and $Y_k^e \subseteq U_j$ for some ϵ_1-regular pair (U_i, U_j), and $|X_k^e|, |Y_k^e| \geq \epsilon^4 \hat{n}/2$. Thus, since $2\epsilon_1/\epsilon^4 \geq \epsilon/2$, the pair (X_k^e, Y_k^e) is $(\epsilon/2)$-regular. In particular, its density cannot be smaller than $1/2$ by more than, say, ϵ, i.e., it must be larger than say $2/5$. The same holds for each pair (Y_0, X_k^e), since $Y_0 \subset U_{i_0}$ and each X_k^e is contained in U_j for some j such that (U_i, U_j) is ϵ_1-regular of density at least $1/2$ in blue. Hence, in particular, the assertion holds for $\alpha \neq 1$.

Now let $\alpha = 1$. Apply the above argument for $\bar{\alpha} = 1 + \epsilon/2$ and replace each set Y_i obtained in this way by a subset Y_i' of Y_i of size $|Y_i'| = \tilde{n} = |X_i|$. Then, for the number of pairs (X_i, Y_i'), we get

$$\tilde{n} = \frac{(1-\epsilon/2)n}{(2+\bar{\alpha})k} = \frac{(1-\epsilon/2)n}{(3+\epsilon/2)\tilde{n}} > \frac{(1-\epsilon)n}{3\tilde{n}} = \frac{1-\epsilon}{(2y+\alpha)\tilde{n}}.$$

Furthermore, each pair (X_i, Y_i) was $(\epsilon/2)$-regular with density at least $2/5$, so each pair (X_i, Y_i') is ϵ-regular with density at least $2/5 - 2\epsilon > 1/3$. ∎

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